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1984 J. Phys. A: Math. Gen. 17 L879

(http://iopscience.iop.org/0305-4470/17/16/004)

J. Phys. A: Math. Gen. 17 (1984) L879–L881. Printed in Great Britain

LETTER TO THE EDITOR

A connection between Van Hove's and Prugovečki's approach to quantisation

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Received 26 July 1984

Abstract. We point out that Prugovečki's recent investigation of a certain phase space quantisation scheme is equivalent to a corresponding scheme, which results from Van Hove's quantisation mapping.

In a recent note Prugovečki (1982) pointed out the stochastic content of a Hilbert space scalar product among functions defined on phase space. His starting point is furnished by the following representation of the quantum operators $\hat{Q}^j$, $\hat{P}^j$, ($h=1$)

$$\hat{Q}^j = q^j + \mathrm{i}\,\partial/\partial p^j, \qquad \hat{P}^j = -\partial/\partial q^j \tag{1}$$

which satisfy the canonical commutation relations (CCR). It becomes obvious from (1) that $\hat{Q}^j$ and $\hat{P}^j$ must act on functions $\psi(p,q)$ which are square integrable on phase space, i.e. they belong to $\mathscr{L}^2(\Gamma)$. To be precise the space Γ is identified with R^6, namely the phase space of a non-relativistic particle. (In the present work we do not consider the relativistic case.)

It turns out that representation (1) is highly reducible so one faces the problem of specifying an appropriate subspace of $L^2(\Gamma)$ that carries an irreducible representation of the CCR. In this way one arrives at a subspace $L^2(\Gamma_\xi)$ specified as follows. Define a resolution generator $\xi \in L^2(\Gamma_\xi)$ by the requirement that $\xi(p,q)$ be rotationally invariant and that the orthogonal projector $P(\Gamma_\xi)$ into $L^2(\Gamma_\xi)$ satisfies the relations

$$P(\Gamma_\xi) = \int_\Gamma |\xi_{q,p}\rangle\,\mathrm{d}p\,\mathrm{d}q\langle\xi_{q,p}| \tag{2}$$

$$\psi_\xi(p,q) \equiv (P(\Gamma_\xi)\psi)(p,q) = \int \xi^*_{q,p}(q',p')\psi(q',p')\,\mathrm{d}q'\,\mathrm{d}p' \tag{3}$$

for $\psi \in L^2(\Gamma)$. In the above relations $\xi_{p,q}$ denotes the translation of ξ, i.e.

$$\xi_{p,q}(p',q') = \mathrm{e}^{\mathrm{i}p(q'-q)}\xi(p'-p,q'-q) \tag{4}$$

In the present letter we intend to study the scalar product in Hilbert space from a viewpoint which differs from that of Prugovečki, in the hope that our approach will enrich the above quantisation scheme.

Consider the scalar product of two functions ψ_ξ and φ_ξ in $L^2(\Gamma_\xi)$. It is given by

$$(\psi_\xi, \varphi_\xi) \equiv (\psi, \varphi)_\xi = \int \psi^*_\xi(q,p)\varphi_\xi(1,p)\,\mathrm{d}p\,\mathrm{d}p. \tag{5}$$

Substituting the relevant expression from (3) we obtain

$$(\psi, \varphi)_\xi = \iiint \xi_{q,p}(q', p')\psi^*(q', p')\xi^*_{q,p}(q'', p'')\psi(q'', p'')\mathrm{d}q\, \mathrm{d}p\, \mathrm{d}q'\, \mathrm{d}p'\, \mathrm{d}q''\, \mathrm{d}p''. \qquad (6)$$

The above equation naturally leads us to consider the integral

$$\int \xi_{p,q}(q', p')\xi^*_{p,q}(q'', p'')\, \mathrm{d}p\, \mathrm{d}q$$

which would represent some kind of completeness relation. Suppose we were to set

$$\int \xi_{p,q}(q', p')\xi^*_{q,p}(q'', p'')\, \mathrm{d}p\, \mathrm{d}q = \delta(p'-p'')\delta(q'-q'').$$

Substituting in (6) we would obtain

$$(\psi, \varphi)_\xi = \int \psi^*(q', p')\psi(q', p')\, \mathrm{d}p'\, \mathrm{d}q'$$

which is clearly unsatisfactory since it returns us to the product of the full space $L^2(\Gamma)$. Clearly, the completeness relation chosen above is unacceptable and must be amended in a suitable way.

Let us introduce a density function $\rho(q, p)$ in phase space and write the completeness relation as follows

$$\int \xi_{(p,q)}(q', p')\xi^*_{q,p}(q'', p'')\, \mathrm{d}q\, \mathrm{d}p = \rho(p', q')\delta(p'-p'')\delta(q'-q'') \qquad (7)$$

Then relation (6) becomes

$$(\psi, \varphi)_\xi \equiv (\psi, \varphi)_\rho = \int \psi^*(q, p)\rho(q, p)\varphi(q, p)\, \mathrm{d}q\, \mathrm{d}p. \qquad (8)$$

What we have achieved is that, instead of projecting ψ and φ into the subspace $L^2(\Gamma_\xi)$ in order to perform the integration in the whole of phase space we have employed a suitable density function. The question is whether a function $\rho(q, p)$ exists which accomplishes this task.

To this end let us note that representation (1) for the quantum operators $\hat{Q}^j$, $\hat{P}^j$ has been given a long time ago by Van Hove (1951). In retrospect Van Hove's quantisation mapping aims at circumventing the incompatibility (Chernoff 1981) of the simultaneous relation of the following three statements ($\hbar = 1$).

(i) $\widehat{\{f, g\}} = -\mathrm{i}[\hat{f}, \hat{g}]$, (ii) $\hat{1} = I$, identity operator,

(iii) $\hat{Q}^j = q^j$, $\hat{P}^j = -\mathrm{i}\, \partial/\partial q^j$

where ^ denotes the quantisation mapping, f and g are functions on phase space and { , } denotes the Poisson bracket.

The aforementioned incompatibility is neatly summarised in the work of Joseph (1970), which in turn is based on a theorem by Wollenberg (1967).

In a recent paper (Ktorides and Papaloucas 1984) we have argued for the need to slighly improve Van Hove's mapping, as follows

$$\hat{Q}^j = \tfrac{1}{2}q^j + \mathrm{i}\, \partial/\partial p^j, \qquad \hat{P}^j = -2\mathrm{i}\, \partial/\partial q^j. \qquad (9)$$

With this improvement we were able to arrive at (8) with a density function uniquely given by the expression

$$\rho(p, q) = e^{-\alpha(p^2+q^2)} \tag{10}$$

where α is a parameter, which can be fixed by a suitable normalisation requirement (usually $\alpha = \frac{1}{2}$).

Surprisingly enough our result coincides with that of Bargmann (1961), which he achieved by discussing coherent states. In Bargmann's case too, one has a mixed q and p representation, which is tantamount to working with a space of states that are phase space functions. Such a description turns out to be over-determined.

It is simply a matter of interpretation whether the correct scalar product in such a Hilbert space is interpreted stochastically, or whether the same result is achieved via the introduction of a density function.

It is our belief that connections between different quantisation schemes such as the one presently made, contribute towards a better understanding of the overall problem. In the present case the usefulness is further pronounced by the fact that Prugovečki (1982) has already extended his considerations to the relativistic domain. It would be of interest if the other point of view (i.e. the one which employs a density function) could be extended to the relativistic domain also.

References

Bargmann V 1961 *Commun. Pure Appl. Math.* XIV 187

Chernoff R R 1981 *Hadronic J.* **4** 879

Joseph A 1970 *Commun. Math. Phys.* **17** 210

Ktorides C N and Papaloucas L C 1984 *A new improved realisation of Van Hove's quantisation mapping and certain consequences, Preprint Athens University* (submitted for publication)

Prugovečki E 1982 *Phys. Rev. Lett.* **40** 1065

Van Hove L 1951 *Proc. R. Acad. Sci. Belgium* **26** 1

Wollenberg S L 1967 *Proc. Am. Math. Soc.* **20** 315